Influence Functions for the Unsteady Surface Element Method

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The unsteady surface element methods provide analytical and numerical procedures for the efficient solution of certain transient heat conduction problems. The methods are based on Duhamel's theorem which requires influence functions. Because the influence functions implicitly include the effects of boundary conditions and geometries, many different ones are needed. Sometimes the most difficult influence functions to evaluate are those at the heated surface and at the smallest dimensionless times. This paper provides some early time influence functions for some basic one-dimensional geometries as well as for some two- and three-dimensional cases for various heated regions on the surface of a semi-infinite body. A given early time solution has the advantage of being valid for a variety of boundary conditions distant from the surface. Some large-time influence functions are also given. An example is given that illustrates that a combination of a short-time solution and a large-time solution can sometimes adequately cover the complete time domain.

Introduction

THE unsteady surface element (USE) method and its variants are compact analytical and numerical methods for the solution of certain transient heat conduction problems.[1,2] They are particularly appropriate for two- and three-dimensional bodies connected over only a portion of the surface. A number of transient influence functions are needed for the effective employment of the unsteady surface element method.

Influence functions are defined in two ways. One involves the heat flow through an area that results from a unit step in temperature at $t=0$. The one discussed in this paper is the average temperature rise over a heated region at time t for a unit heat flux starting at time $t=0$. This function can be used in the deviation of solutions for arbitrary time variations of the surface heat flux.

Influence functions derived ultilizing classical methods are frequently in the form of infinite series that can require many terms for accurate evaluations. Other solutions are given only as integrals. These evaluation difficulties are frequently acute for small times and certain parts of the heated surface. Simple expressions given in this paper for early times for a variety of basic geometries obviate the use of expressions that are expensive to evaluate and difficult to manipulate mathematically. Some long-time expressions are also given.

Early time expressions have greater applicability than "long"-time solutions because the same solution is valid for a number of conditions at boundaries distant from the heat surface.

Another fortunate circumstance for some cases is that the influence function can accurately approximate the complete time domain by using just two simple expressions, one for early times and one for late times.[1,2]

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Influence Functions for Basic One-Dimensional Geometries

Small-Time Approximations

The one-dimensional heat flux influence function $\phi(r,t)$ is the solution of the transient heat conduction equation for a heat flux q_0 equal to unity and starting at time t equal to zero; in symbols, $\phi(r,t)$ is related to the temperature rise, $T(r,t)-T_0$, by

$$\phi(r,t) = [T(r,t) - T_0]/q_0 \tag{1}$$

For both brevity and clarity, the basic cases are denoted herein using the notations given in Refs. 3 and 4, which utilize Green's functions to develop the influence functions. The rectangular coordinates are denoted by X, radial cylindrical by R, and radial spherical by RS. A "1" is used to denote boundary conditions of the first kind (temperature) and a "2" denotes boundary conditions of the second kind (q). If there is no physical boundary, a "0" is used. In this section, the cases considered are a semi-infinite body (X20), plate (X21 and X22), solid cylinder (R02), region bounded by a cylindrical hole (R20), annulus (R22), solid sphere (RS02), and region bounded by a spherical hole (RS20).

The heated surface is the one of primary interest. The exact Laplace transforms of $\phi(r,t)$ at the heated surface, denoted $\tilde{\phi}(s)$, are given for most of these cases in Table 1. The Laplace transforms are given because they are used directly[1,2] by some forms of the USE method and also because they are useful for obtaining small-time expressions. For each case, the heated surface $\tilde{\phi}(s)$ is directly proportional to $(k\rho cs^3)^{-1/2}$, where k is thermal conductivity, ρ the density, c the specific heat, and s the Laplace transform parameter. For that reason, $\tilde{\phi}(s)(k\rho cs^3)^{1/2}$ is tabulated in Table 1; it is a function of the dimensionless transform variable,

$$v = (sa^2/\alpha)^{1/2} \tag{2}$$

where a is the thickness of the plate or radius of the cylinder or sphere. For the X-- cases, heating is at $x=0$ and for the radial cases at $r=a$.

For the annulus (R22), $\tilde{\phi}(s)\,(k\rho cs^3)^{1/2}$ is given by

$$\tilde{\phi}(s)\,(k\rho cs^3)^{1/2} = \frac{I_0(v)K_1(vb^+) + I_1(vb^+)K_0(v)}{I_1(vb^+)K_1(v) - I_1(v)K_1(vb^+)}(-1)^{\beta} \tag{3}$$

where the $r=b$ surface is insulated and $b^+ \equiv b/a$. The exponent β is zero for $b^+ \geq 1$ and unity for $b^+ < 1$.

Small times correspond to large values of s, and thus v, in the Laplace domain. For that reason, the expressions in Table 1 and Eq. (3) are examined for large v; each case can be approximated by[5]

$$\tilde{\phi}(s) \approx (k\rho cs^3)^{-1/2}\left(1 + \frac{B_1}{v} + \frac{B_2}{v^2} + \frac{B_3}{v^3} + \frac{B_4}{v^4} + De^{-2v|b^+-1|}\right) \tag{4}$$

where the B_i and D coefficients are given in Table 1. For cases X22 and X21, replace $|b^+-1|$ by unity in the exponent in Eq. (4).

A number of comments are made about Table 1 and Eq. (4) to emphasize that the coefficients have a pattern. The very early time behavior is similar to that of a uniformly heated semi-infinite body. The plate cases 1b and 1c are similar, with $\pm 2\exp(-2v)$ being a correction for the boundary condition at $x=$a. The coefficients B_i in Table 1 have several striking similarities. The same set of coefficients (except for sign) are found for the solid cylinder and for the region outside a cylindrical hole. The same is true for the two spherical cases. The coefficients of the spherical cases are constant in magnitude and are almost twice as large as those for the cylindrical cases. All the coefficients in Table 1 are linear corrections for curvature effects, but B_1 is the dominant term for small times.

For the two-plate cases $\tilde{\phi}(s)\ (k\rho cs^3)^{1/2}$ is plotted in Fig. 1 vs

$$\tau \equiv \alpha/a^2 s \tag{5}$$

which is used because small values of τ correspond to small times. Inspection of Fig. 1 reveals that $\tilde{\phi}(s)\,(k\rho cs^3)^{1/2}$ is nearly unity until $\tau \approx 0.2$, which can be related to the limiting small time for plates, $t^+ \equiv \alpha t/a^2 \approx 0.2$. Until this time, the insulation or isothermal conditions at $x=a$ do not affect the temperature at $x=0$. This boundary effect is described by the D coefficient term in Eq. (4).

For the cylindrical cases $\tilde{\phi}(s)\,(k\rho cs^3)^{1/2}$ is plotted in Fig. 2. Again, it approaches unity in all cases for small τ values. For the solid cylinder R02, it is always unity or larger, while for R20 it is always unity or less. The τ value at which both deviate from unity by about 2% is ~ 0.002. For τ values less than 0.002, the surface temperature rises as if the body were semi-infinite. For an annulus R22, $\tilde{\phi}(s)$ deviates from the two basic cases of R02 and R20: if $b^+ \to 0$, the R02 result is obtained and if $b^+ \to \infty$, R20 results. The cylindrical void case R20 is particularly difficult to approximate for the complete time domain.

Notice that the annular cases of Table 1 have linear corrections for both the curvature and finite thickness effects, i.e., B_1 and D are not zero.

The time domain expression for small times for the temperature rise is given by

$$T - T_0 \approx 2q_0\left(\frac{t}{\pi k\rho c}\right)^{1/2} + \frac{q_0 a}{k}[C_1 t^+ + C_2(t^+)^{3/2} + C_3(t^+)^2 + C_4(t^+)^{5/2}] + 2D\frac{q_0 a}{k}(t^+)^{1/2}\mathrm{ierfc}[(t^+)^{1/2}] \tag{6}$$

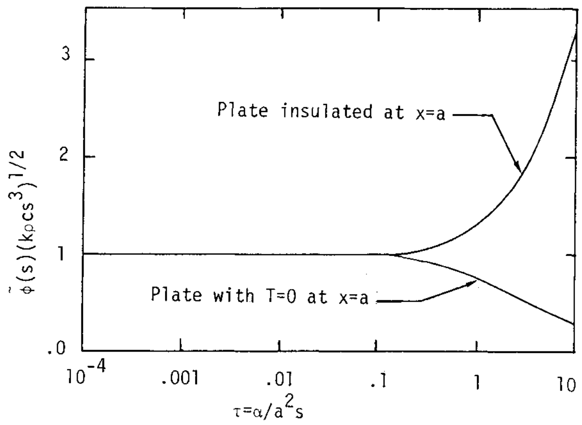

Fig. 1 $\tilde{\theta}(s)\ (k\rho cs^3)^{1/2}$ **for plate cases.**

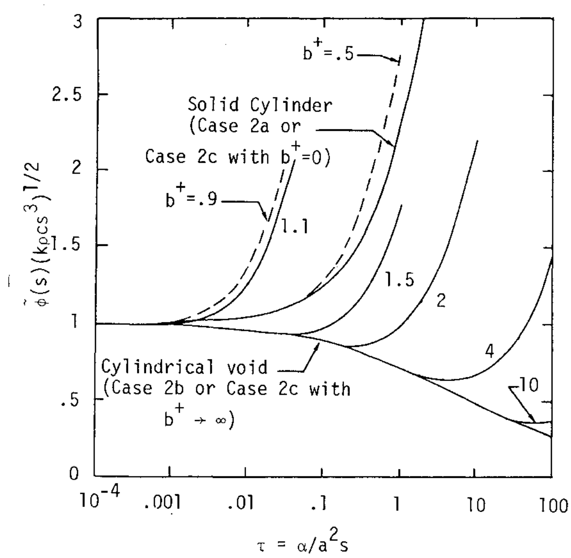

Fig. 2 $\tilde{\theta}(s)\ (k\rho cs^3)^{1/2}$ **for cylinder cases.**

Table 1 Laplace transforms and coefficients for small times of some basic one-dimensional cases

	Case	Ref.	$\tilde{\phi}(s)\,(k\rho cs^3)^{1/2}$	B_1	B_2	B_3	B_4	D
1a	X20	6, p.305	1	0	0	0	0	0
1b	X22	6, p.310	coth(v)	0	0	0	0	2
1c	X21		tanh(v)	0	0	0	0	−2
2a	R02	6, p.329	$I_0(v)/I_1(v)$	½	⅜	⅜	63/128	0
2b	R20	6, p.338	$K_0(v)/K_1(v)$	−½	⅜	−⅜	63/128	0
2c	R22($b<a$)	6, p.332	Eq. (3)	½				2
2d	R22(b$>a$)	6, p.332	Eq. (3)	−½				2
3a	RS02	7, p.183	$v/[v\ \coth(v)-1]$	1	1	1	1	0
3b	RS20		$v/(1+v)$	−1	1	−1	1	0

Table 2 Coefficients for large times of some basic cases

	Case	Ref.	F	F_2	F_3	G_1	G_2	G_3
lb	X22	6, p.112	1/3	1	π			
lc	X21	6, p.113	1	0	$\pi/2$			
2a	R02	6, p.203	1/4	2	3.8317			
3a	RS02		1/5	3	4.4934			
3b	RS20	6, p.248				1	1	½
s-i	circle	11,12				$8/3\pi$	½	$\pi/24$
s-i	rect	9				Eq.(14)	$2b^+/\pi$	$b^+(1+b^{+2})/9$

where t^+ is the dimensionless time, $t^+ = \alpha t/a^2$. The C_i values in Eq. (6) are related to the B_i values by

$$C_1 = B_1, \quad C_2 = 4B_2/3\pi^{1/2}, \quad C_3 = B_3/2, \quad C_4 = (8/15\pi^{1/2})B_4 \tag{7}$$

The $|C_i|$ values decrease monotonically with i. Notice also that the coefficients C_i are equal to or less than the respective B_i values. Consequently, if a B_i term in the Laplace domain equation (4) is small, its relative effect is even smaller in the time domain equation (6). This is important because the Laplace domain expressions are simpler and easier to evaluate than the real-time expressions.

Large-Time Expressions for Finite Geometries

Expressions for the temperature rise for large times for "finite" geometries exposed to a constant heat flux can be written as

$$T - T_0 = \frac{q_0 a}{k}\left(F_1 + F_2 t^+ - \frac{2}{F_3^2} e^{-F_3^2 t^+}\right) \tag{8}$$

Table 2 gives the value of F_1, F_2, and F_3. The unlike geometries of a plate (X22), solid cylinder (R02), and sphere (RS02) have F_i values varying in a systematic way such as the F_2 values being 1, 2, and 3, respectively. (Klamkin[13] investigated a more general problem of arbitrary bodies, but did not give specific results.) For an annulus heated at $r=a$ and insulated at $r=b$ (R22 case), F_1 and F_2 are

$$F_1 = \frac{1}{1-b^{+2}}\left[\frac{1}{4} - b^{+2}\left(\frac{3}{4} + \frac{b^{+2}}{1-b^{+2}} \ell n b^+\right)\right]$$

$$F_2 = \frac{2}{1-b^{+2}} \tag{9}$$

For the region outside the cylinder R20, the temperature at $r=a$ is[8]

$$T - T_0 = \frac{q_0 a}{2k}\left\{\left(\ell n \frac{4t^+}{C}\right)\left[1 + \frac{1}{2t^+} - \frac{1}{16t^{+2}}\left(1 + 3\ell n\frac{4t^+}{C}\right)\right] + \frac{1}{2t^+} + \frac{1}{32t^{+2}}(\pi^2+3)\right\},$$

$$\ell n C = 0.5772\ldots = \text{Euler's const} \tag{10}$$

In a number of cases, the short-time solution given by Eq. (6) and the large-time solution by Eq. (8) can be used to span the complete time domain. An example is X22 for which the expressions agree to five significant figures at $t^+ = 0.25$. (Not all cases agree so well!)

Heated Region on a Semi-infinite Body

Small-Time Approximations for Heated Regions on a Semi-infinite Body

Small-time approximations for the average temperature over heated regions on the surface of a semi-infinite body are given here. The heat flux q_0 is constant over the region and the surface is insulated elsewhere. Five basic heating shapes are considered: 1) half of a semi-infinite surface, 2) quarter of a semi-infinite surface, 3) infinite strip of width $2a$, 4) rectangle $2a \times 2b$, and 5) circle of radius a. For the first case, the average is given for the heated region of width a next to the edge. For the second case, the average is given for the region $a \times a$ at the corner. For all five cases, the average temperature can be given by

$$\bar{T} - T_0 \approx 2q_0\left(\frac{t}{\pi k \rho c}\right)^{1/2} + \frac{q_0 a}{k}\left[-\frac{H_1}{\pi}t^+ + \frac{2H_2}{3}\left(\frac{t^+}{\pi}\right)^{3/2} + \frac{H_3}{4\pi}t^{+2}\right] \tag{11}$$

The coefficients H_i are given in Table 3. As for the cases given in Table 1, the leading term is for a uniformly heated semi-infinite body.

The H_1 term in Eq. (11) is an "edge" correction and is given by

$$H_1 = a\frac{\text{unheated perimeter}}{\text{heated region being averaged}} \tag{12}$$

which is quite general and applies to other regions such as triangles.

Table 3 Coefficients for small times for heated regions at the surface of a semi-infinite body

Heated region	Ref.	H_1	H_2	H_3
1) Half	6,p. 264	1	0	0
2) Quarter		2	1	0
3) Strip, $2a$ wide	9	1	0	0
4) Rectangle	9	$1+(b^+)^{-1}$	$1/b^+$	0
5) Circle	11,12	2	0	1

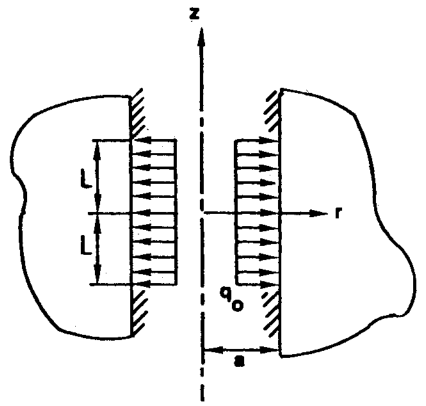

Fig. 3 Problem of region exterior to cylinder with heating over a finite length.

The H_2 term is a "corner" correction. The H_3 term is a curvature correction for the circular region.

Large-Time Approximations

Large-time expressions for a number of different infinite bodies heated over a finite region can be given by

$$\frac{\bar{T}-T_0}{q_0 a/k}=G_1-\frac{G_2}{(\pi t^+)^{1/2}}+\frac{G_3}{(\pi t^+)^{3/2}} \tag{13}$$

and the G_i are given in Table 2 for three cases. The first case is for the region outside a sphere (RS20). The last two cases are for semi-infinite bodies with a circular surface heat source of radius a and a rectangular source, $2a\times 2b$ in dimensions. For the last two cases, which are labeled $s=i$ circle and s-i rect, the average temperature $\bar{T}$ over the heated region is given. The sources are constant heat fluxes over the indicated area. For the last case, G_1 is

$$G_1=\frac{2}{\pi}\Big\{\ln[b^+ + (b^{+2}+1)^{1/2}] + b^+\ln\Big[\frac{1}{b^+}+\Big(\frac{1}{b^{+2}}+1\Big)^{1/2}\Big] + \frac{1}{3b^+}[1+b^{+3}-(b^{+2}+1)^{3/2}]\Big\} \tag{14}$$

which for a square source ($b^+=1$) is equal to 0.9464. Again there are similarities even though there are two different geometries. Each case has large-time behavior, with the G_i being independent of time. The numerical G_1 values for the last three cases of Table 2 are 1, 0.8488, and 0.9464 (for $b^+=1$), respectively. An approximate relation for G_1 for the heated surfaces on a semi-infinite body (last two cases of Table 2) is

$$G_1\approx\frac{8}{3\pi}\frac{(A/\pi)^{1/2}}{a}=0.4789\frac{A^{1/2}}{a} \tag{15a}$$

where A is the heated area. For the circular region, this expression is exact; for the square, it gives a result that is 1.2% high. For the geometry outside a sphere, G_1 is given by

$$G_1=\frac{(A/2\pi)^{1/2}}{a}=0.3989\frac{A^{1/2}}{a} \tag{15b}$$

where A is the heated area of a hemispherical cavity on the surface of a semi-infinite body.

The second term in Eq. (13) for all cases can be exactly given by the strikingly simple expression,

$$G_2=\frac{\text{heated area in semi-infinite region}}{2\pi a^2} \tag{16}$$

Since Eq. (16) is true for the disparate cases, it is expected that it would be true also for other heated regions such as outside a cube and for a triangular source on a semi-infinite body. A relation for G_3 that is valid for the last two cases of Table 3 is

$$G_3=\frac{(\text{heated area})\times(\text{moment of inertia about center})}{12a^4} \tag{17}$$

Additivity of Solutions

For all the small-time solutions examined herein, the various corrections (edge, corner, and curvature) are simple additive terms. For this reason, it is postulated that other solutions can be constructed by simple addition. One such case is for the region outside a cylindrical hole of radius a, which has the same geometry as case 2b of Table 1. In this present case, however, heating is uniform only over the finite length of $2L$ and insulated elsewhere as shown in Fig. 3. The average temperature for small times of the heated region is first considered.

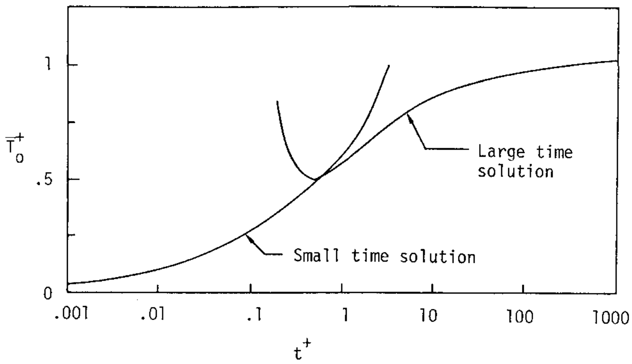

Fig. 4 Average temperature rise for heated region shown in Fig. 3.

The problem contains two previously considered, basic geometric elements: an infinite strip heated on a semi-infinite body and the region outside a cylindrical hole. For the uniformly heated strip of width $2L$ on a semi-infinite body, the average temperature rise is obtained from case 3 of Table 3,

$$\frac{\bar{T}-T_0}{q_0 a/k}\approx 2\Big(\frac{\alpha t}{a^2\pi}\Big)^{1/2}-\frac{1}{\pi}\frac{\alpha t}{aL} \tag{18}$$

The temperature rise for the cylindrical hole is found from Eq. (6) and the coefficients are found from case 2b of Table 1 and Eq. (7)

$$\frac{\bar{T}-T_0}{q_0 a/k}\approx 2\Big(\frac{\alpha t}{a^2\pi}\Big)^{1/2}-\frac{1}{2}\frac{\alpha t}{a^2}+\frac{1}{2\pi^{1/2}}\frac{(\alpha t)^{3/2}}{a^3} \tag{19}$$

The leading term in both Eqs. (18) and (19) is $2(\alpha t/a^2\pi)^{1/2}$. The principle of additivity is to add the corrections to the leading term; the dimensionless result is

$$\frac{\bar{T}-T_0}{q_0 a/k}\approx 2\Big(\frac{t^+}{\pi}\Big)^{1/2}-\Big(\frac{1}{\pi}\frac{a}{L}+\frac{1}{2}\Big)t^+ +\frac{1}{2\pi^{1/2}}(t^+)^{3/2} \tag{20}$$

Notice that the result obtained from the additivity principle is *not* obtained by the usual superposition procedure. A graph of Eq. (20) for $L/a=1$ is given in Fig. 4 as the curve on the left.

The large-time transient behavior is now considered; it is dominated by the boundary conditions distant from the heated surface. The form of the solution should be the same as Eq. (13), which is for a uniformly heated finite region inside a semi-infinite body. The constant G_1 is the steady-state value that can be approximated by Eq. (15b) and is accurately given in Ref. 10; G_2 is given by Eq. (16),

$$G_2=\pi a 2L/2\pi a^2=L/a \tag{21}$$

Since the heated region is not a flat surface on a semi-infinite body, Eq. (17) is not appropriate for G_3. Instead, the heated surface area, $2\pi a_c 2L$, is made equal to that of the RS20 case, $4\pi a_s^2$, where a_c is the radius of Fig. 3 and a_s is the spherical radius. Then, using $a_s^2=a_c L$ in the third term of Eq. (13), $a_c\to a$, and $G_3=1/2$ for RS20 from Table 2 gives the G_3 value $(L/a)^2/2$. Using these G_2 and G_3 expressions in Eq. (13) gives

$$\frac{k\bar{T}_s}{q_0 a}\approx G_1-\frac{L/a}{(\pi t^+)^{1/2}}+\frac{(L/a)^2}{2(\pi t^+)^{3/2}} \tag{22}$$

For $L/a=1$, the exact value of G_1 is 1.0300, while Eq. (15b) gives 1.0. The large-time curve of Fig. 4 is obtained from Eq. (22) using $L/a=1$ and $G_1=1.03$; the small- and large-time solutions match very well with the small-time solution accurate for $t^+ < 0.5$ and the large-time solution for $t^+ > 0.5$. This problem illustrates that a short-time solution given by Eq. (20) and a larger-time solution given by Eq. (22) can be used to span the complete time domain. Thus, seemingly complicated problems can sometimes be easily solved using the influence functions given in this paper.

Acknowledgment

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References

[1]Keltner, N. R. and Beck, J. V., "Unsteady Surface Element Method," *Journal of Heat Transfer*, Vol. 103, 1981, pp. 759-764.

[2]Beck, J. V. and Keltner, N. R., "Transient Thermal Contact of Two Semi-infinite Bodies Over a Circular Area," *AIAA Progress in Astronautics and Aeronautics: Spacecraft Radiative Transfer and Temperature Control,* Vol. 83,edited by T. E. Horton, AIAA, New York, 1982, pp. 61-82.

[3]Beck, J. V., "Green's Function Solution for Transient Heat Conduction Problems," *International Journal of Heat and Mass Transfer*, Vol. 27, 1984, pp. 1235-1244.

[4]Beck, J. V., "Green's Functions and Numbering System for Transient Conduction," *AIAA Journal*, to be published.

[5]Abramowitz, M. and Stegun, I. A. (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards, Applied Mathematics Series 55, Government Printing Office, Washington, D.C., 1964, pp. 377-378.

[6]Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, 2nd ed., Oxford University Press, London, 1959.

[7]Luikov, A. V., *Analytical Heat Diffusion Theory*, edited by J. P. Hartnett, Academic Press, New York, 1968.

[8]Cooper, L. Y., "Heating of a Cylindrical Cavity," *International Journal of Heat and Mass Transfer*, Vol. 19, 1976, pp. 575-577.

[9]Keltner, N. R., Bainbridge, B. L., and Beck, J. V., "Rectangular Heat Source on a Semi-infinite Solid—An Analysis for a Thin Film Heat Flux Gage Calibration," ASME Paper 84-HT-46, 1984.

[10]Beck, J. V., Yen, D. H. Y., and Johnson, B., "Steady State Temperature Distribution for Infinite Region Outside a Partially Heated Cylinder," ASME Paper 82-HT-24 and *Chemical Engineering Communications*, Vol. 26, Nos. 4-6, 1984, pp. 355-367.

[11]Beck, J. V., "Large Time Solutions for Temperatures in a Semi-infinite Body with a Disk Heat Source," *International Journal of Heat and Mass Transfer*, Vol. 24, 1981, pp. 155-164.

[12]Beck, J. V., "Average Transient Temperature within a Body Heated by a Disk Heat Source," *AIAA Progress in Astronautics and Aeronautics: Heat Transfer, Thermal Control and Heat Pipes*, Vol. 70, edited by W. B. Olstad, AIAA, New York, 1980, pp. 3-24.

[13]Klamkin, M. S., "Asymptotic Heat Conduction in Arbitrary Bodies," Phillips Research Repts., Vol. 30, Nos. 31*-39*, 1975.